More on crinkles in the last scattering surface

Rishi Khatri*

Department of Astronomy, University of Illinois at Urbana-Champaign,
1002 W. Green Street, Urbana, IL 61801 USA

Benjamin D. Wandelt†

Department of Physics, University of Illinois at Urbana-Champaign,
1110 W. Green Street, Urbana, IL 61801 USA
Department of Astronomy, University of Illinois at Urbana-Champaign,
1002 W. Green Street, Urbana, IL 61801 USA and
California Institute of Technology, Mail Code 130-33, Pasadena, CA 91125 USA

(Dated: May 13, 2010)

Abstract

Inhomogeneous recombination can give rise to perturbations in the electron number density which can be a factor of five larger than the perturbations in baryon density. We do a thorough analysis of the second order anisotropies generated in the cosmic microwave background (CMB) due to perturbations in the electron number density. We show that solving the second order Boltzmann equation for photons is equivalent to solving the first + second order Boltzmann equations and then taking the second order part of the solution. We find the approximate solution to the photon Boltzmann hierarchy in ℓ modes and show that the contributions from inhomogeneous recombination to the second order monopole, dipole and quadrupole are numerically small. We also point out that perturbing the electron number density in the first order tight coupling and damping solutions for the monopole, dipole and quadrupole is not equivalent to solving the second order Boltzmann equations for inhomogeneous recombination. Finally we confirm our result in a previous paper that inhomogeneous recombination gives rise to a local type non-Gaussianity parameter $f_{NL} \sim -1$. The signal to noise for the detection of the temperature bispectrum generated by inhomogeneous recombination is ~ 1 for an ideal full sky experiment measuring modes up to $\ell_{max} = 2500$.

* rkhatri2@illinois.edu

† bwandelt@illinois.edu

I. INTRODUCTION

The process of recombination depends on the energy density of photons and baryons as well as the number density of electrons. Perturbations in energy and number density of photons, baryons and electrons therefore makes recombination a function of position. The resulting perturbations in the electron number density, δ_e, give rise to second order perturbations in the photons through Compton scattering. The perturbations in the electron number density were first calculated by Novosyadlyj [1], who found that $\delta_e \sim 5 \times \delta_b$ on large scales, where δ_b is the perturbation in the baryon density. Recently Senatore et al. [2] did a more rigorous analysis, including perturbations in the escape probability of Lyα photons, and found a similar result.

The factor of five enhancement of the electron number perturbation suggests the possibility of observable non-Gaussianity even if the initial conditions are completely Gaussian. Assessing whether these effects are observable by Planck [3] is therefore important, especially since Planck aims to probe the non-Gaussianities in the initial conditions. There have been many studies of different second order effects [4–21]. In our previous paper [22] (hereafter KW09) we calculated the bispectrum arising due to inhomogeneous recombination and found that it gives rise to a local type non-Gaussianity with the non linear (NL) parameter $|f_{NL}| \lesssim 1$. However we ignored the second order photon monopole and quadrupole and electron velocity in the second order Boltzmann equation. In this paper we justify ignoring these terms. We also examine two different methods of arriving at the second order solutions to the photon Boltzmann equation. The first method is to solve the first and second order Boltzmann equations together and take the second order part of the resulting solution as the solution to the second order Boltzmann equation. The second method is to solve the second order Boltzmann equation separately. In KW09 we solved the second order Boltzmann equation separately and found that the first order photon monopole does not contribute to the second order anisotropy while the first order photon dipole is partially cancelled by the first order electron velocity. We prove that the two methods are equivalent. This is also important for the self-consistency of the perturbation theory. The important fact that the first order source terms are suppressed is somewhat obscured in the expression resulting from solving the first and second order equations together. We also explain in the conclusions section that perturbing the number density of electrons in the first order tight coupling and

damping solutions for the monopole, dipole and quadrupole is not equivalent to solving the second order Boltzmann equation for inhomogeneous recombination. The method of perturbing the first order solutions was followed in [23] whereas what we want is the solution to the second order Boltzmann equation which we find in this paper. Following cosmological parameters are used for numerical calculations: baryon density $\Omega_b = 0.0418$, cold dark matter density $\Omega_c = 0.1965$, cosmological constant $\Omega_\Lambda = 0.7617$, number of massless neutrinos $N_\nu = 3.04$, Hubble constant $H_0 = 73$, CMB temperature $T_{CMB} = 2.725$, primordial Helium fraction $y_{He} = 0.24$, spectral index of the primordial power spectrum $n_s = 1.0$, and $\sigma_8 = 0.8$. All first order quantities are in conformal Newtonian gauge and calculated using CMBFAST [24]. Electron number density perturbation is calculated using DRECFAST [1].

II. LINE OF SIGHT INTEGRATION AT SECOND ORDER: METHOD 1

We begin with the first + second order equations as given in, for example, [25] Equations 6.6 and 6.11. We drop the second order metric perturbations and products of first order terms which do not contain δ_e, the electron number density perturbation. However we retain the full first order equation since it gives rise to second order terms, as we will later see. We drop the usual factors of 1/2 multiplied with the second order variables, and use $\Theta^{(i)} \equiv \Delta^{(i)}/4$ as our perturbation variable for convenience. $\Theta \equiv \delta T/T$ is the photon temperature perturbation while Δ is the perturbation in the photon distribution function integrated over momentum and normalized appropriately [25]. Superscripts (i) denote the order of perturbation. In what follows all perturbation variables are functions of coordinates on spatial hypersurface $\mathbf{x}$, line of sight angle $\hat{\mathbf{n}}$ and conformal time η in real space and functions of Fourier mode $\mathbf{k}$, $\hat{\mathbf{n}}$ and η in Fourier space unless specified otherwise. We will use same symbols for real space and Fourier space quantities but that should not cause any confusion as only one quantity is needed at a time. Boldface quantities are 3-vectors while $\hat{}$ indicates a unit 3-vector. We use the following metric signature with $\phi = \phi^{(1)} + \phi^{(2)} + ...$ etc. and ignoring vector and tensor modes

$$ds^2 = a^2(\eta)\left[-e^{2\psi}d\eta^2 + e^{-2\phi}dx^2\right] \tag{1}$$

Also we decompose the first order temperature perturbation in Fourier space into ℓ modes as $\Theta^{(1)}(\eta, \mathbf{k}, \hat{\mathbf{n}}) = \sum_\ell (-i^\ell)(2\ell+1)P_\ell(\hat{\mathbf{n}}.\hat{\mathbf{k}})\Theta^{(1)}_\ell(\eta, \mathbf{k})$, where $P_\ell(\hat{\mathbf{n}}.\hat{\mathbf{k}})$ are the Legendre poly-

nomials. For the second order temperature perturbation we use the spherical harmonic decomposition defined by, $\Theta^{(2)}_{\ell m}(\eta, \mathbf{x}) = \int d\hat{\mathbf{n}} \Theta^{(2)}(\eta, \mathbf{x}, \hat{\mathbf{n}}) Y^*_{\ell m}(\hat{\mathbf{n}})$ and similarly in Fourier space. Note that this differs from the convention used in [25] by a factor of $(-i)^{-\ell}\sqrt{\frac{2\ell+1}{4\pi}}$. Also the electron velocity, $\mathbf{v_e}^{(1)}$, is equal to the baryon velocity to a high precision and we will drop the subscript on $\mathbf{v_e}$ in the rest of the paper.

We start with the first + second order Boltzmann equation for photons in real space, ignoring second order metric perturbations and second order terms which are products of first order terms but do not contain $\delta_e \equiv (n_e - \bar{n}_e)/\bar{n}_e$, where $n_e(\eta, \mathbf{x})$ is the electron number density and $\bar{n}_e(\eta)$ is the mean electron number density.

$$\begin{aligned}
&\frac{d}{d\eta}\left[\Theta^{(1)}(\eta, \mathbf{x}, \hat{\mathbf{n}}) + \psi^{(1)}(\eta, \mathbf{x}) + \Theta^{(2)}(\eta, \mathbf{x}, \hat{\mathbf{n}})\right] - \frac{\partial}{\partial \eta}(\phi^{(1)}(\eta, \mathbf{x}) + \psi^{(1)}(\eta, \mathbf{x})) = \\
&\bar{n}_e(\eta)\sigma_T a(\eta)\left[\left(1 + \delta^{(1)}_e(\eta, \mathbf{x})\right)\left(C^{(1)}(\eta, \mathbf{x}, \hat{\mathbf{n}}) - \Theta^{(1)}(\eta, \mathbf{x}, \hat{\mathbf{n}}) - \Theta^{(2)}(\eta, \mathbf{x}, \hat{\mathbf{n}})\right)\right. \\
&\left.+\frac{1}{\sqrt{4\pi}}\Theta^{(2)}_{00}(\eta, \mathbf{x}) + \frac{1}{10}\sum_m \Theta^{(2)}_{2m}(\eta, \mathbf{x}) Y_{2m}(\hat{\mathbf{n}}) + \mathbf{v}^{(2)}(\eta, \mathbf{x}).\hat{\mathbf{n}}\right],
\end{aligned} \tag{2}$$

where we have defined $C^{(1)}$ which is given in Fourier space by

$$C^{(1)}(\eta, \mathbf{k}, \hat{\mathbf{n}}) \equiv \Theta^{(1)}_0(\eta, \mathbf{k}) - \frac{1}{2}\Theta^{(1)}_2(\eta, \mathbf{k}) P_2(\hat{\mathbf{k}}.\hat{\mathbf{n}}) + \mathbf{v}^{(1)}(\eta, \mathbf{k}).\hat{\mathbf{n}}, \tag{3}$$

$\frac{d}{d\eta}$ denotes the total derivative which is equal to $\frac{\partial}{\partial\eta} + n^i\frac{d}{dx^i}$ along the line of sight to zeroth order. $\hat{n}$ denotes the line of sight direction, σ_T is the Thomson scattering cross section. We now add $\bar{n}_e\sigma_T a(1 + \delta^{(1)}_e)\psi^{(1)}$ to Equation 2. Doing this and rearranging terms we get,

$$\begin{aligned}
&\left[\frac{d}{d\eta} - \dot{\tau}\left(1 + \delta^{(1)}_e\right)\right]\left[\Theta^{(1)} + \psi^{(1)} + \Theta^{(2)}\right] = R(\eta, \mathbf{x}, \hat{\mathbf{n}}), \\
&R(\eta, \mathbf{x}, \hat{\mathbf{n}}) \equiv \frac{\partial}{\partial\eta}(\phi^{(1)} + \psi^{(1)}) - \dot{\tau}\left[\left(1 + \delta^{(1)}_e\right)\left(C^{(1)} + \psi^{(1)}\right) + \frac{1}{\sqrt{4\pi}}\Theta^{(2)}_{00} + \frac{1}{10}\sum_m \Theta^{(2)}_{2m} Y_{2m}(\hat{\mathbf{n}}) + \mathbf{v}^{(2)}.\hat{\mathbf{n}}\right],
\end{aligned} \tag{4}$$

where we have defined $\dot{\tau}(\eta) \equiv -\bar{n}_e\sigma_T a$, with $\tau(\eta) = -\int_\eta^{\eta_0} \dot{\tau} d\eta$. η_0 is the conformal time at $a = 1$. Now we use the fact that along the photon geodesic $\mathbf{x}$ is a function of η to write Equation 4 as

$$e^{-\int_\eta^{\eta_0} d\eta' \dot{\tau}\left(1+\delta^{(1)}_e\right)|_{\mathbf{x}(\eta')}} \frac{d}{d\eta}\left[\left(\Theta^{(1)} + \psi^{(1)} + \Theta^{(2)}\right) e^{\int_\eta^{\eta_0} d\eta' \dot{\tau}\left(1+\delta^{(1)}_e\right)|_{\mathbf{x}(\eta')}}\right] = R(\eta, \mathbf{x}, \hat{\mathbf{n}}) \tag{5}$$

Note that the above equation can only be written if the integrals appearing are evaluated along the line of sight and so $\mathbf{x}$ ceases to be an independent variable outside the integrals.

Integrating Equation 5 formally along the line of sight results in

$$\left(\Theta^{(1)} + \psi^{(1)} + \Theta^{(2)}\right)|_{\mathbf{x}(\eta_0)}(\eta_0) = \int_0^{\eta_0} d\eta e^{\int_\eta^{\eta_0} d\eta' \dot\tau \left(1+\delta_e^{(1)}\right)|_{\mathbf{x}(\eta')}} \left[R(\eta, \mathbf{x}, \hat{\mathbf{n}})\right]_{\mathbf{x}(\eta)}$$
$$= \int_0^{\eta_0} d\eta e^{-\tau}\left(1 + \int_\eta^{\eta_0} d\eta' \dot\tau \delta_e^{(1)}|_{\mathbf{x}(\eta')}\right)\left[R(\eta, \mathbf{x}, \hat{\mathbf{n}})\right]_{\mathbf{x}(\eta)} \qquad (6)$$

In the last line we have assumed that $\int_\eta^{\eta_0} d\eta' \dot\tau \delta_e^{(1)}|_{\mathbf{x}(\eta')}$ is small compared to unity and approximately of same order as $\delta_e^{(1)}$, which is a good enough assumption once recombination starts.

Taking the second order part of the above equation we get

$$\Theta^{(2)}|_{\mathbf{x}(\eta_0)}(\eta_0) =$$
$$\int_0^{\eta_0} d\eta e^{-\tau}\left[(-\dot\tau)\left\{\delta_e^{(1)}\left(C^{(1)} + \psi^{(1)}\right) + \frac{\Theta_{00}^{(2)}}{\sqrt{4\pi}} + \frac{1}{10}\sum_m \Theta_{2m}^{(2)} Y_{2m}(\hat{\mathbf{n}}) + \mathbf{v}^{(2)}.\hat{\mathbf{n}}\right\}\right.$$
$$\left.+\left\{\int_\eta^{\eta_0} d\eta' \dot\tau \delta_e^{(1)}|_{\mathbf{x}(\eta')}\right\}\left\{\frac{\partial}{\partial\eta}(\phi^{(1)} + \psi^{(1)}) - \dot\tau\left(C^{(1)} + \psi^{(1)}\right)\right\}\right]_{\mathbf{x}(\eta)}. \qquad (7)$$

If we consider a single observer then we don't have an independent three dimensional space variable with respect to which we can Fourier transform this equation. If we consider all possible observers then $\mathbf{y} \equiv \mathbf{x}(\eta_0)$ spans all space at time η_0 and we can write $\mathbf{x}(\eta) = \mathbf{x}_0 + \hat{\mathbf{n}}\eta = \mathbf{y} + \hat{\mathbf{n}}(\eta - \eta_0)$ along the line of sight. Now all quantities in Equation 7 are functions of the same variable $\mathbf{y}$ and we can take Fourier transform with respect to it. The result is (Note that all perturbation variables are Fourier transforms of the respective quantities in the rest of this section, we omit the arguments $(\mathbf{k})$ where there is no confusion.)

$$\Theta^{(2)}(\eta_0, \mathbf{k}, \hat{\mathbf{n}}) =$$
$$\int_0^{\eta_0} d\eta e^{i\mathbf{k}.\hat{\mathbf{n}}(\eta-\eta_0)} e^{-\tau(\eta)}\left[(-\dot\tau(\eta))\left\{\left(\int \frac{d^3k'}{(2\pi)^3}\delta_e^{(1)}(\mathbf{k}', \eta)\left(C^{(1)}(\mathbf{k}-\mathbf{k}', \eta) + \psi^{(1)}(\mathbf{k}-\mathbf{k}', \eta)\right)\right)\right.\right.$$
$$\left.+\frac{\Theta_{00}^{(2)}(\eta, \mathbf{k})}{\sqrt{4\pi}} + \frac{1}{10}\sum_m \Theta_{2m}^{(2)}(\eta, \mathbf{k}) Y_{2m}(\hat{\mathbf{n}}) + \mathbf{v}^{(2)}(\eta, \mathbf{k}).\hat{\mathbf{n}}\right\}$$
$$+\left\{\int \frac{d^3k'}{(2\pi)^3}\int_\eta^{\eta_0} d\eta' e^{i\mathbf{k}'.\hat{\mathbf{n}}(\eta'-\eta)}\dot\tau(\eta')\delta_e^{(1)}(\mathbf{k}', \eta')\right\}$$
$$\left.\times\left\{\frac{\partial}{\partial\eta}(\phi^{(1)}(\mathbf{k}-\mathbf{k}', \eta) + \psi^{(1)}(\mathbf{k}-\mathbf{k}', \eta)) - \dot\tau(\eta)\left(C^{(1)}(\mathbf{k}-\mathbf{k}', \eta) + \psi^{(1)}(\mathbf{k}-\mathbf{k}', \eta)\right)\right\}\right], \qquad (8)$$

where we have used the properties of Fourier transform when the variable getting transformed is shifted and which gives the phase factors on the right hand side. We could also

have chosen initial point $\mathbf{x_0} = \mathbf{y}'$ or $\mathbf{x}(\eta_1) = \mathbf{y_1}$ as our integration variable for any fixed η_1 and got the same result.

III. LINE OF SIGHT INTEGRATION AT SECOND ORDER: METHOD 2

Another way to do the formal integration of the Boltzmann equation is to move all terms containing $\delta_e^{(1)}$ and primordial potentials to the right hand side in Equation 2, take Fourier transform of the resulting equation and then integrate along the line of sight. This is in fact what is done in [25] and KW09. In that case the solution for $\Theta^{(2)}$ is

$$\Theta^{(2)}(\eta_0,\mathbf{k}) = \int_0^{\eta_0} d\eta e^{i\mathbf{k}.(\mathbf{x}(\eta)-\mathbf{x}(\eta_0))}e^{-\tau}\left[(-\dot{\tau})\left\{\left(\int\frac{d^3k'}{(2\pi)^3}\delta_e^{(1)}(\mathbf{k}')\left(C^{(1)}(\mathbf{k}-\mathbf{k}')-\Theta^{(1)}(\mathbf{k}-\mathbf{k}')\right)\right)\right.\right.$$
$$\left.\left.+\frac{\Theta_{00}^{(2)}}{\sqrt{4\pi}}+\frac{1}{10}\sum_m\Theta_{2m}^{(2)}Y_{2m}(\hat{\mathbf{n}})+\mathbf{v}^{(2)}.\hat{\mathbf{n}}\right\}\right] \tag{9}$$

We now integrate by parts in variable η the term involving $\Theta^{(1)}$. The boundary terms vanish, resulting in

$$\int_0^{\eta_0} d\eta e^{i\mathbf{k}.(\mathbf{x}(\eta)-\mathbf{x}(\eta_0))}e^{-\tau}\dot{\tau}\left(\int\frac{d^3k'}{(2\pi)^3}\delta_e^{(1)}(\mathbf{k}')\Theta^{(1)}(\mathbf{k}-\mathbf{k}')\right)$$
$$=\int\frac{d^3k'}{(2\pi)^3}e^{-i\mathbf{k}.\mathbf{x}(\eta_0)}\int_0^{\eta_0}d\eta e^{i\mathbf{k}'.\mathbf{x}(\eta)}\dot{\tau}\delta_e^{(1)}(\mathbf{k}')\left(e^{-\tau}e^{i(\mathbf{k}-\mathbf{k}').\mathbf{x}(\eta)}\Theta^{(1)}(\mathbf{k}-\mathbf{k}')\right)$$
$$=\int\frac{d^3k'}{(2\pi)^3}e^{-i\mathbf{k}.\mathbf{x}(\eta_0)}\int_0^{\eta_0}d\eta\left\{\int_\eta^{\eta_0}d\eta' e^{i\mathbf{k}'.\mathbf{x}(\eta')}\dot{\tau}(\eta')\delta_e^{(1)}(\mathbf{k}',\eta')\right\}\frac{d}{d\eta}\left(e^{-\tau}e^{i(\mathbf{k}-\mathbf{k}').\mathbf{x}(\eta)}\Theta^{(1)}(\mathbf{k}-\mathbf{k}')\right) \tag{10}$$

We now use the first order equation for $\Theta^{(1)}$ to obtain

$$\int\frac{d^3k'}{(2\pi)^3}e^{-i\mathbf{k}.\mathbf{x}(\eta_0)}\int_0^{\eta_0}d\eta\left\{\int_\eta^{\eta_0}d\eta' e^{i\mathbf{k}'.\mathbf{x}(\eta')}\dot{\tau}(\eta')\delta_e^{(1)}(\mathbf{k}',\eta')\right\}e^{-\tau}e^{i(\mathbf{k}-\mathbf{k}').\mathbf{x}(\eta)}$$
$$\times\left(-\dot{\tau}C^{(1)}(\mathbf{k}-\mathbf{k}')-i(\mathbf{k}-\mathbf{k}').\hat{\mathbf{n}}\psi^{(1)}(\mathbf{k}-\mathbf{k}')+\frac{\partial\phi(\mathbf{k}-\mathbf{k}')}{\partial\eta}\right)$$
$$=\int\frac{d^3k'}{(2\pi)^3}e^{i\mathbf{k}.(\mathbf{x}(\eta)-\mathbf{x}(\eta_0))}\int_0^{\eta_0}d\eta\left\{\int_\eta^{\eta_0}d\eta' e^{i\mathbf{k}'.(\mathbf{x}(\eta')-\mathbf{x}(\eta))}\dot{\tau}(\eta')\delta_e^{(1)}(\mathbf{k}',\eta')\right\}e^{-\tau}$$
$$\times\left(-\dot{\tau}C^{(1)}(\mathbf{k}-\mathbf{k}')-i(\mathbf{k}-\mathbf{k}').\hat{\mathbf{n}}\psi^{(1)}(\mathbf{k}-\mathbf{k}')+\frac{\partial\phi(\mathbf{k}-\mathbf{k}')}{\partial\eta}\right) \tag{11}$$

By doing integration by parts once again on terms containing ψ in Equation 11, similar to what is done in solving the first order Boltzmann equation [26], and then using the result in Equation 9, we obtain Equation 8. This shows the simple connection between the two approaches.

In KW09 we worked with Equation 9. In Equation 9 it is readily apparent that there is cancellation between the collision term $C^{(1)}$ and $\Theta^{(1)}$. This point is somewhat obscured in Equation 8 since the cancellation is now happening between δ_e terms. Nevertheless we have shown the exact equivalence of the two approaches and that there is cancellation of first order terms which leads to a small value of f_{NL} even though the electron number density is enhanced by a factor of ~ 5. It is also clear from Equation 9 that the term which causes the cancellation, $\delta_e^{(1)}\Theta^{(1)}$, has no direct counterpart among the source terms in the first order Boltzmann equation. Thus we have to be careful while using analogies with the first order Boltzmann equation to estimate the second order solutions. We will return to this point in the conclusions section.

IV. BOLTZMANN HIERARCHY AT SECOND ORDER

The Boltzmann equation for photons in Fourier space, ignoring all the first order terms that do not involve the electron number density perturbation is [25]

$$\begin{aligned}
&\dot{\Theta}^{(2)}(\mathbf{k},\hat{\mathbf{n}},\eta) + i\hat{\mathbf{n}}.\mathbf{k}\Theta^{(2)}(\mathbf{k},\hat{\mathbf{n}},\eta) - \dot{\tau}\Theta^{(2)}(\mathbf{k},\hat{\mathbf{n}},\eta) = S^{(2)}(\mathbf{k},\hat{\mathbf{n}},\eta)\\
&S^{(2)}(\mathbf{k},\hat{\mathbf{n}},\eta) \equiv -\dot{\tau}\int\frac{d^3k'}{(2\pi)^3}\delta_e^{(1)}(\mathbf{k}-\mathbf{k}',\eta)\left[\Theta_0^{(1)}(\mathbf{k}',\eta) - \sum_{\ell''}(-i)^{\ell''}(2\ell''+1)P_{\ell''}(\hat{\mathbf{n}}.\hat{\mathbf{k}}')\Theta_{\ell''}^{(1)}(\mathbf{k}',\eta)\right.\\
&\qquad\left.+\hat{\mathbf{n}}.\hat{\mathbf{k}}'v^{(1)}(\mathbf{k}',\eta) - \frac{1}{2}P_2(\hat{\mathbf{k}}'.\hat{\mathbf{n}})\Pi^{(1)}(\mathbf{k}',\eta)\right]\\
&\qquad-\dot{\tau}\left[\frac{\Theta_{00}^{(2)}}{\sqrt{4\pi}}(\mathbf{k},\eta) + \frac{1}{10}\sum_{m'}\Theta_{2m'}^{(2)}(\mathbf{k},\eta)Y_{2m'}(\hat{\mathbf{n}}) + \mathbf{v}^{(\mathbf{2})}(\mathbf{k},\eta).\hat{\mathbf{n}}\right]\\
&\quad= -\dot{\tau}\int\frac{d^3k'}{(2\pi)^3}\delta_e^{(1)}(\mathbf{k}-\mathbf{k}',\eta)\left[-\sum_{\ell''\geq 2}(-i)^{\ell''}(2\ell''+1)P_{\ell''}(\hat{\mathbf{n}}.\hat{\mathbf{k}}')\Theta_{\ell''}^{(1)}(\mathbf{k}',\eta)\right.\\
&\qquad\left.+\hat{\mathbf{n}}.\left(\hat{\mathbf{k}}'v^{(1)}(\mathbf{k}',\eta) - \mathbf{V}_\gamma^{(1)}(\mathbf{k}',\eta)\right) - \frac{1}{2}P_2(\hat{\mathbf{k}}'.\hat{\mathbf{n}})\Pi^{(1)}(\mathbf{k}',\eta)\right]\\
&\qquad-\dot{\tau}\left[\frac{\Theta_{00}^{(2)}}{\sqrt{4\pi}}(\mathbf{k},\eta) + \frac{1}{10}\sum_{m'}\Theta_{2m'}^{(2)}(\mathbf{k},\eta)Y_{2m'}(\hat{\mathbf{n}}) + \mathbf{v}^{(\mathbf{2})}(\mathbf{k},\eta).\hat{\mathbf{n}}\right],
\end{aligned} \tag{12}$$

where $\mathbf{V}_\gamma^{(1)}$ is the first order photon velocity. $\mathbf{V}_\gamma^{(1)}$ and $\mathbf{V}_\gamma^{(2)}$, the second order photon

velocity are defined as follows [25]:

$$
\begin{aligned}
(\rho_\gamma + p_\gamma)\,\mathbf{V}_\gamma &= \int \frac{d^3p}{(2\pi)^3} f\mathbf{p}, \\
\mathbf{V}_\gamma{}^{(1)}(\mathbf{k'}) &= \frac{3}{4\pi}\int d\hat{\mathbf{n}}\Theta^{(1)}(\mathbf{k'},\eta,\hat{\mathbf{n}})\hat{\mathbf{n}}, \\
\mathbf{V}_\gamma^{(\mathbf{2})}(\mathbf{k},\eta) &= \frac{3}{4\pi}\int d\hat{\mathbf{n}}\Theta^{(2)}(\mathbf{k},\eta,\hat{\mathbf{n}})\hat{\mathbf{n}} - 4\int \frac{d^3k'}{(2\pi)^3}\Theta_0^{(1)}(\mathbf{k}-\mathbf{k'},\eta)\mathbf{V}_\gamma^{(\mathbf{1})}(\mathbf{k'},\eta) \\
&\approx \frac{3}{4\pi}\int d\hat{\mathbf{n}}\Theta^{(2)}(\mathbf{k},\eta,\hat{\mathbf{n}})\hat{\mathbf{n}}
\end{aligned} \tag{13}
$$

In the last line we have ignored the second term since it does not contain $\delta_e^{(1)}$. We remark that this extra term in the above equation partially cancels a term of the form $\Theta_0^{(1)} \times v$ in the full second order equation. The dot product of photon velocities with line of sight direction which appears in the Boltzmann equation is given by

$$
\begin{aligned}
\mathbf{V}_\gamma{}^{(1)}(\mathbf{k'}).\hat{\mathbf{n}} &= -i\Theta_1^{(1)}(\mathbf{k'},\eta)4\pi\sum_{m'} Y_{1m'}^*(\hat{\mathbf{k}}')Y_{1m'}(\hat{\mathbf{n}}) \\
\mathbf{V}_\gamma^{(\mathbf{2})}(\mathbf{k},\eta).\hat{\mathbf{n}} &= \sum_{m'}\Theta_{1m'}^{(2)}(\mathbf{k},\eta)Y_{1m'}(\hat{\mathbf{n}}).
\end{aligned} \tag{14}
$$

We choose $\hat{\mathbf{z}}$ axis along $\hat{\mathbf{k}}$ and take the spherical harmonic transform of Equation 12

$$
\begin{aligned}
\dot{\Theta}_{\ell m}^{(2)} &= \dot{\tau}\Theta_{\ell m}^{(2)} - ik\left[\sqrt{\frac{(\ell-m)(\ell+m)}{(2\ell-1)(2\ell+1)}}\Theta_{\ell-1m}^{(2)} + \sqrt{\frac{(\ell+1-m)(\ell+1+m)}{(2\ell+3)(2\ell+1)}}\Theta_{\ell+1m}^{(2)}\right] + S_{\ell m}^{(2)}, \\
S_{\ell m}^{(2)} &= -\dot{\tau}\int \frac{d^3k'}{(2\pi)^3}\delta_e^{(1)}(\mathbf{k}-\mathbf{k'},\eta)\left[-(1-\delta_{\ell 0})(1-\delta_{\ell 1})4\pi(-i)^\ell\Theta_\ell^{(1)}(\mathbf{k'},\eta)Y_{\ell m}^*(\hat{\mathbf{k}}')\right. \\
&\quad \left. -\frac{1}{2}\frac{4\pi}{5}Y_{2m}^*(\hat{\mathbf{k}}')\delta_{\ell 2}\Pi^{(1)}(\mathbf{k'},\eta)\right] - \dot{\tau}\left[\Theta_{00}^{(2)}\delta_{\ell 0}\delta_{m0} + \frac{1}{10}\Theta_{2m}^{(2)}\delta_{\ell 2} + V_m^{(2)}\delta_{\ell 1} + S_{\delta v}^m\delta_{\ell 1}\right].
\end{aligned} \tag{15}
$$

In above we have defined

$$
S_{\delta v}^m \equiv \int \frac{d^3k'}{(2\pi)^3}\delta_e^{(1)}(\mathbf{k}-\mathbf{k'},\eta)\left[\frac{4\pi}{3}Y_{1m}^*(\hat{\mathbf{k}}')\left(v^{(1)}(\mathbf{k'},\eta) + 3i\Theta_1^{(1)}(\mathbf{k'},\eta)\right)\right] \tag{16}
$$

and $V_m^{(2)}\delta_{\ell 1}$ is the spherical harmonic transform of $\mathbf{v}^{(2)}.\hat{\mathbf{n}}$. All second order quantities are functions of $(\mathbf{k},\eta)$. Note that different m modes are independent of each other. Now we can write down the Boltzmann hierarchy explicitly.

$$
\dot{\Theta}_{00}^{(2)} = -\frac{ik}{\sqrt{3}}\Theta_{10}^{(2)}
$$

$$
\dot{\Theta}_{1m}^{(2)} = -ik\left[\sqrt{\frac{1}{3}}\Theta_{00}^{(2)}\delta_{m0} + \sqrt{\frac{4-m^2}{15}}\Theta_{2m}^{(2)}\right] - \dot{\tau}\left[V_m^{(2)} - \Theta_{1m}^{(2)} + S_{\delta v}^m\right] \tag{17}
$$

$$
\dot{\Theta}_{2m}^{(2)} = -ik\left[\sqrt{\frac{4-m^2}{15}}\Theta_{1m}^{(2)} + \sqrt{\frac{9-m^2}{35}}\Theta_{3m}^{(2)}\right] + \frac{9\dot{\tau}}{10}\Theta_{2m}^{(2)} - \dot{\tau}S_{\delta 2}^m \tag{18}
$$

For $\ell \geq 3$,

$$\dot{\Theta}^{(2)}_{\ell m} = \dot{\tau}\Theta^{(2)}_{\ell m} - ik\left[\sqrt{\frac{(\ell-m)(\ell+m)}{(2\ell-1)(2\ell+1)}}\Theta^{(2)}_{\ell-1m} + \sqrt{\frac{(\ell+1-m)(\ell+1+m)}{(2\ell+3)(2\ell+1)}}\Theta^{(2)}_{\ell+1m}\right] - \dot{\tau}S^m_{\delta\ell}$$
$$S^m_{\delta 2} \equiv \int \frac{d^3k'}{(2\pi)^3}\delta^{(1)}_e(\mathbf{k}-\mathbf{k}',\eta)\left[4\pi\Theta^{(1)}_2(\mathbf{k}',\eta)Y^*_{2m}(\hat{\mathbf{k}}') - \frac{4\pi}{10}Y^*_{2m}(\hat{\mathbf{k}}')\Pi^{(1)}(\mathbf{k}',\eta)\right]$$
$$S^m_{\delta\ell} \equiv \int \frac{d^3k'}{(2\pi)^3}\delta^{(1)}_e(\mathbf{k}-\mathbf{k}',\eta)\left[-4\pi(-i)^\ell\Theta^{(1)}_\ell(\mathbf{k}',\eta)Y^*_{\ell m}(\hat{\mathbf{k}}')\right] \tag{19}$$

We note that the first order monopole does not appear in the above equations. Also the first order photon dipole is partially cancelled by the first order electron dipole. Thus only the first order quadrupole and higher multipoles contribute to the hierarchy. These first order terms are small during recombination and thus we should expect the second order terms due to inhomogeneous recombination to be small. This cancellation counteracts the production of non-Gaussianity due to enhancement in $\delta^{(1)}_e$.

V. APPROXIMATE SOLUTION OF BOLTZMANN HIERARCHY

To find the approximate solutions we can use the fact that during recombination $\dot{\tau} >> 1/\eta$. Then, as in the case of the first order Boltzmann equation, we can attempt to find an approximate solution at different orders in $1/\dot{\tau}$. In the limit of $\dot{\tau} >> 1/\eta$, which is true during the entire recombination period except at the very end when the visibility also drops sharply, we can ignore the $\ell \geq 3$ modes. Also in Equation 18 we can ignore terms with $\ell \geq 2$ which do not involve $\dot{\tau}$. Equation 18 with these approximations is

$$\Theta^{(2)}_{2m} = \frac{10ik}{9\dot{\tau}}\sqrt{\frac{4-m^2}{15}}\Theta^{(2)}_{1m} + \frac{10}{9}S^m_{\delta 2} \tag{20}$$

Using this in Equation 17,

$$\dot{\Theta}^{(2)}_{1m} = -ik\sqrt{\frac{1}{3}}\Theta^{(2)}_{00}\delta_{m0} + \frac{2(4-m^2)k^2}{27\dot{\tau}}\Theta^{(2)}_{1m} - \frac{10ik}{9}\sqrt{\frac{4-m^2}{15}}S^m_{\delta 2}$$
$$-\dot{\tau}\left[V^{(2)}_m - \Theta^{(2)}_{1m} + S^m_{\delta v}\right]. \tag{21}$$

To proceed further we need the momentum equation for baryons [27]. Note that we ignore the second order metric perturbations and the terms arising from the first order perturbations that do not contain $\delta^{(1)}_e$ as we did with the Boltzmann equation for photons [2, 25].

$$
\begin{aligned}
\frac{\partial \mathbf{v}^{(2)}}{\partial \eta} &= -\mathcal{H}\mathbf{v}^{(2)} + \frac{\dot{\tau}}{R}\left[\int \frac{d^3k'}{(2\pi)^3}\delta_e^{(1)}(\mathbf{k}-\mathbf{k}',\eta)\left(\mathbf{v}^{(\mathbf{1})}(\mathbf{k}',\eta) - \mathbf{V}_\gamma^{(\mathbf{1})}(\mathbf{k}',\eta)\right)\right. \\
&\quad \left. + \left(\mathbf{v}^{(2)}(\mathbf{k},\eta) - \mathbf{V}_\gamma^{(2)}(\mathbf{k},\eta)\right)\right] \\
&\approx \frac{\dot{\tau}}{R}\left[\int \frac{d^3k'}{(2\pi)^3}\delta_e^{(1)}(\mathbf{k}-\mathbf{k}',\eta)\left(\mathbf{v}^{(\mathbf{1})}(\mathbf{k}',\eta) - \mathbf{V}_\gamma^{(\mathbf{1})}(\mathbf{k}',\eta)\right)\right. \\
&\quad \left. + \left(\mathbf{v}^{(2)}(\mathbf{k},\eta) - \mathbf{V}_\gamma^{(2)}(\mathbf{k},\eta)\right)\right] \qquad (22)
\end{aligned}
$$

We have defined ratio of mean baryon to mean photon density $R \equiv 3\bar{\rho}_b/4\bar{\rho}_\gamma$. Ignoring the expansion term above introduces only a small error on small scales (factors of $(1+R)^{1/4}$) which is not important here (for example see Chap 8, Exercise 5 in [26], also [28]). We take the dot product of above equation with line of sight direction $\hat{\mathbf{n}}$ and take the spherical harmonic transform of the resulting equation. The result is

$$
\frac{\partial V_m^{(2)}}{\partial \eta} = \frac{\dot{\tau}}{R}\left[S_{\delta v}^m + V_m^{(2)} - \Theta_{1m}^{(2)}\right] \qquad (23)
$$

We can expand Equation 23 perturbatively in $R/\dot{\tau}$ as in the first order case [26, 28]. At zeroth order in $R/\dot{\tau}$ all the source terms (terms which are products of the first order terms) vanish. This causes all the intrinsic second order terms to also vanish if we impose Gaussian initial conditions. Thus all terms in the hierarchy are of first order or higher in $R/\dot{\tau}$. At first order in $R/\dot{\tau}$ we have

$$
V_m^{(2)} = \Theta_{1m}^{(2)} - S_{\delta v}^m \qquad (24)
$$

Using this in Equation 23 we get up to second order in $\frac{R}{\dot{\tau}}$

$$
V_m^{(2)} = \Theta_{1m}^{(2)} - S_{\delta v}^m + \frac{R}{\dot{\tau}}\frac{\partial}{\partial \eta}\left(\Theta_{1m}^{(2)} - S_{\delta v}^m\right) \qquad (25)
$$

Continuing like this we can obtain the terms at higher orders in $\frac{R}{\dot{\tau}}$. Note that in first order perturbation theory we need to go to second order in factors of $\frac{R}{\dot{\tau}}$ to get the damping solution. However here we are interested in the contribution of δ_e to the second order anisotropies which are intrinsically of first order in $\frac{R}{\dot{\tau}}$ and it suffices to work at first order in visible factors of $\frac{R}{\dot{\tau}}$. This gives us the leading term in the solution of the second order Boltzmann equation. We comment on the solution beyond this approximation in Appendix B. At leading order in $\frac{R}{\dot{\tau}}$ the equations simplify a lot and the solution is similar to that of the first order Boltzmann equation [28]. Using Equation 25 in Equation 21 we get (dropping a higher order term from

Equation 21)

$$\dot{\Theta}^{(2)}_{1m} = -\frac{ik}{1+R}\sqrt{\frac{1}{3}}\Theta^{(2)}_{00}\delta_{m0} - \frac{10ik}{9(1+R)}\sqrt{\frac{4-m^2}{15}}S^m_{\delta 2} + \frac{R}{1+R}\frac{\partial S^m_{\delta v}}{\partial \eta} \tag{26}$$

$$\begin{aligned}\ddot{\Theta}^{(2)}_{00} &= -\frac{ik}{\sqrt{3}}\dot{\Theta}^{(2)}_{10} \\ &= -k^2c_s^2\Theta^{(2)}_{00} - \frac{4\sqrt{5}}{9}k^2c_s^2S^0_{\delta 2} - ikR\sqrt{3}c_s^2\frac{\partial S^0_{\delta v}}{\partial \eta}\end{aligned} \tag{27}$$

The solution to this equation in the limit that the sound speed $c_s \equiv \sqrt{1/3(1+R)}$ is slowly varying is given by

$$\begin{aligned}\Theta^{(2)}_{00} &= C_1\sin\left[kr_s(\eta)\right] + C_2\cos\left[kr_s(\eta)\right] \\ &\quad - \int_0^\eta d\eta'\left[\frac{4\sqrt{5}}{9}k^2c_s^2(\eta')S^0_{\delta 2}(\eta') + ikR(\eta')\sqrt{3}c_s^2(\eta')\frac{\partial S^0_{\delta v}}{\partial \eta}(\eta')\right]\frac{\sin\left[k(r_s(\eta)-r_s(\eta'))\right]}{kc_s(\eta')},\end{aligned} \tag{28}$$

where we have defined the sound horizon $r_s(\eta) \equiv \int_0^\eta d\eta' c_s(\eta')$. With the Gaussian initial conditions, the second order part of temperature anisotropy and its derivative are initially zero. Thus $C_1 = C_2 = 0$. Integrating by parts the $S_{\delta v}$ term we get, assuming slowly varying c_s,

$$\begin{aligned}\Theta^{(2)}_{00} &= -\int_0^\eta d\eta'\left[\frac{4\sqrt{5}}{9}kc_s(\eta')S^0_{\delta 2}(\eta')\right]\sin\left[k(r_s(\eta)-r_s(\eta'))\right] \\ &\quad - \int_0^\eta d\eta'\left[iR(\eta')\sqrt{3}kc_s^2(\eta')S^0_{\delta v}(\eta')\right]\cos\left[k(r_s(\eta)-r_s(\eta'))\right]\end{aligned} \tag{29}$$

Taking derivative with respect to η of above equation we get

$$\begin{aligned}\Theta^{(2)}_{10} &= \frac{i\sqrt{3}}{k}\dot{\Theta}^{(2)}_{00} \\ &= -\int_0^\eta d\eta'\left[\frac{4i\sqrt{15}}{9}kc_s(\eta')c_s(\eta)S^0_{\delta 2}(\eta')\right]\cos\left[k(r_s(\eta)-r_s(\eta'))\right] \\ &\quad + R(\eta)3c_s^2(\eta)S^0_{\delta v}(\eta) - \int_0^\eta d\eta'\left[R(\eta')3kc_s^2(\eta')c_s(\eta)S^0_{\delta v}(\eta')\right]\sin\left[k(r_s(\eta)-r_s(\eta'))\right]\end{aligned} \tag{30}$$

For $m = \pm 1$ modes we can directly integrate Equation 26.

$$\Theta^{(2)}_{1m=\pm 1}(\eta) = -\int_0^\eta d\eta'\frac{10ik}{9(1+R(\eta'))}\sqrt{\frac{4}{15}}S^m_{\delta 2}(\eta') + \frac{R}{1+R}S^m_{\delta v} \tag{31}$$

We can combine Equations 30 and 31 to get

$$\begin{aligned}\Theta_{1m}^{(2)} &= -\int_0^\eta d\eta' \left[\frac{4i\sqrt{15}}{9} k c_s(\eta') c_s(\eta) S_{\delta 2}^0(\eta')\right] \cos\left[k(r_s(\eta) - r_s(\eta'))\right] \delta_{m0} \\ &\quad - \int_0^\eta d\eta' \left[R(\eta') 3k c_s^2(\eta') c_s(\eta) S_{\delta v}^0(\eta')\right] \sin\left[k(r_s(\eta) - r_s(\eta'))\right] \delta_{m0} \\ &\quad - \int_0^\eta d\eta' \frac{10ik}{9(1+R(\eta'))} \sqrt{\frac{4}{15}} S_{\delta 2}^m(\eta')(1-\delta_{m0}) + \frac{R}{1+R} S_{\delta v}^m \end{aligned} \tag{32}$$

The quadrupole is given by ignoring the $1/\dot{(\tau)}$ term in Equation 20 (at the level of approximation we are working).

$$\Theta_{2m}^{(2)} = \frac{10}{9} S_{\delta 2}^m \tag{33}$$

Finally the second order baryon velocity is given by (Equation 24)

$$\begin{aligned}V_m^{(2)} &= \Theta_{1m}^{(2)} - S_{\delta v}^m \\ &= -\int_0^\eta d\eta' \left[\frac{4i\sqrt{15}}{9} k c_s(\eta') c_s(\eta) S_{\delta 2}^0(\eta')\right] \cos\left[k(r_s(\eta) - r_s(\eta'))\right] \delta_{m0} \\ &\quad - \int_0^\eta d\eta' \left[R(\eta') 3k c_s^2(\eta') c_s(\eta) S_{\delta v}^0(\eta')\right] \sin\left[k(r_s(\eta) - r_s(\eta'))\right] \delta_{m0} \\ &\quad - \int_0^\eta d\eta' \frac{10ik}{9(1+R(\eta'))} \sqrt{\frac{4}{15}} S_{\delta 2}^m(\eta')(1-\delta_{m0}) - \frac{1}{1+R} S_{\delta v}^m \end{aligned} \tag{34}$$

An important point to note here is that the photon and baryon velocities are not equal. In particular the sign of the last term above is different (in addition to a factor of R). These were assumed to be equal in [23].

VI. NUMERICAL RESULTS

We want to calculate the angular averaged bispectrum due to $\Theta_{00}^{(2)}$, $V_m^{(2)}$ and $\Theta_{2m}^{(2)}$. The contribution from $\Theta_{00}^{(2)}$ as well as the $S_{\delta 2}$ terms in $V_m^{(2)}$ to the angular averaged bispectrum is exactly zero. This is shown in Appendix A. The reason that the contribution from $\Theta_{00}^{(2)}$ vanishes is the absence of first order monopole from the second order Boltzmann equations. The contribution to $\Theta_{00}^{(2)}$ from the first order dipole and quadrupole averages to zero. Same is true for the contribution from first order quadrupole terms in $V_m^{(2)}$.

Thus the only terms which will give non-zero contribution to the angular averaged bispectrum are $\Theta_{2m}^{(2)}$ and $S_{\delta v}$ terms in $V_m^{(2)}$. $\Theta_{2m}^{(2)}$ and the last term in Equation 34 are same as the terms already calculated in KW09 with additional multiplying factors. The integral

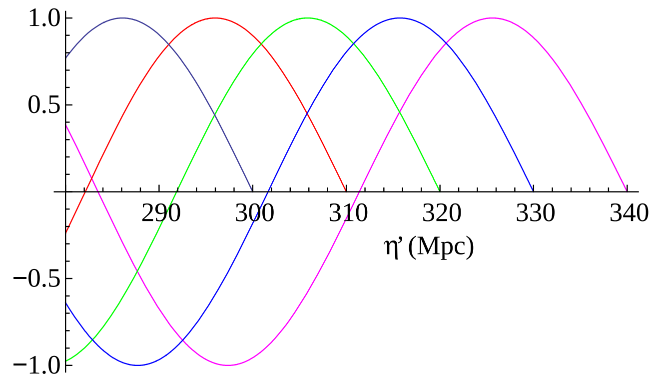

FIG. 1. $\sin\left[k(r_s(\eta) - r_s(\eta'))\right]$ for $k = 0.25$ as a function of η' for different values of η. All curves end at $\eta' = \eta$

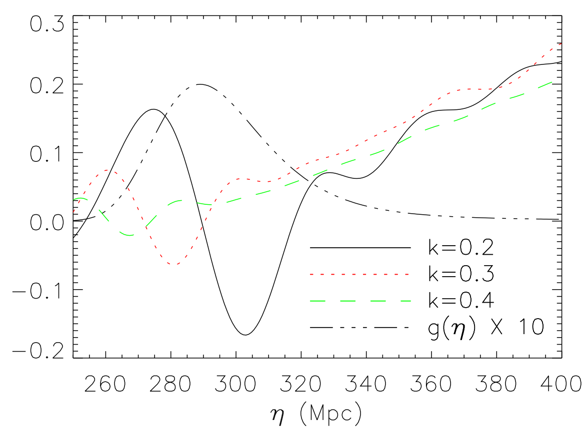

FIG. 2. $3\Theta_1^{(1)} - iv^{(1)}$ as a function of conformal time η for wavenumber $k = 0.2, 0.3, 0.4\,\mathrm{Mpc}^{-1}$. Note that it becomes almost monotonically increasing at large η when photon free streaming becomes important.

term involving $S_{\delta v}$ in Equation 34 can be calculated exactly following the calculation in Appendix A. However there is an easier way to estimate the magnitude of this term. Figure 1 shows the function $\sin\left[k(r_s(\eta) - r_s(\eta'))\right]$ at $k = 0.25$ for different values of η as a function of η'. In general there will be cancellation due to oscillations in the $\sin\left[k(r_s(\eta) - r_s(\eta'))\right]$ as well as $S_{\delta v}$ (Figure 2 and [1, 2]). We can get an upper bound for the region after the peak of the visibility function when the magnitude of $3\Theta_1^{(1)} - iv^{(1)}$ is monotonically increasing by assuming that the last half cycle of the sine contributes without any cancellation and $S_{\delta v}(\eta') \sim S_{\delta v}(\eta)$. Thus we arrive at the following approximation (with slowly varying sound

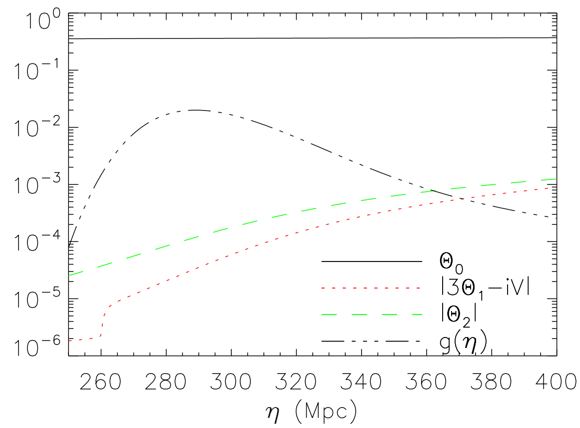

FIG. 3. $\Theta_0^{(1)}$, $|3\Theta_1^{(1)} - iv^{(1)}|$ and $|\Theta_2^{(1)}|$ as a function of η for wavenumber $k = 0.001\,\mathrm{Mpc}^{-1}$. Also shown is the visibility function $g(\eta) \equiv -\dot{\tau} e^{-\tau}$.

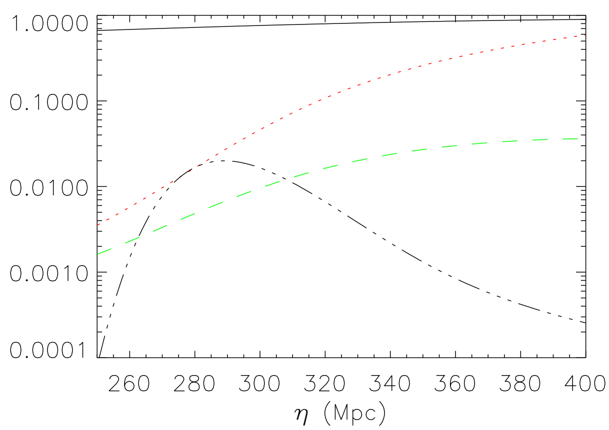

FIG. 4. $\Theta_0^{(1)}$, $|3\Theta_1^{(1)} - iv^{(1)}|$ and $|\Theta_2^{(1)}|$ as a function of η for wavenumber $k = 0.01\,\mathrm{Mpc}^{-1}$. The key is the same as in Figure 3.

speed assumption)

$$
\begin{aligned}
& - \int_0^\eta d\eta' \left[R(\eta') 3k c_s^2(\eta') c_s(\eta) S^0_{\delta v}(\eta') \right] \sin\left[k(r_s(\eta) - r_s(\eta')) \right] \delta_{m0} \\
\lesssim & - \left[R(\eta) 3 c_s^2(\eta) S^0_{\delta v}(\eta) \right] \int_{kr_s(\eta)-\pi}^{kr_s(\eta)} d\left[k r_s(\eta') \right] \sin\left[k(r_s(\eta) - r_s(\eta')) \right] \delta_{m0} \\
= & - \frac{2R(\eta)}{1+R(\eta)} S^0_{\delta v}(\eta) \delta_{m0},
\end{aligned}
\tag{35}
$$

where the $\lesssim$ sign is understood to be with respect to the magnitude of the terms. For most values of η and k, where we don't have a monotonic $3\Theta_1^{(1)} - iv^{(1)}$, there will be additional cancellations due to the oscillations in $3\Theta_1^{(1)} - iv^{(1)}$. Thus the above term will be smaller

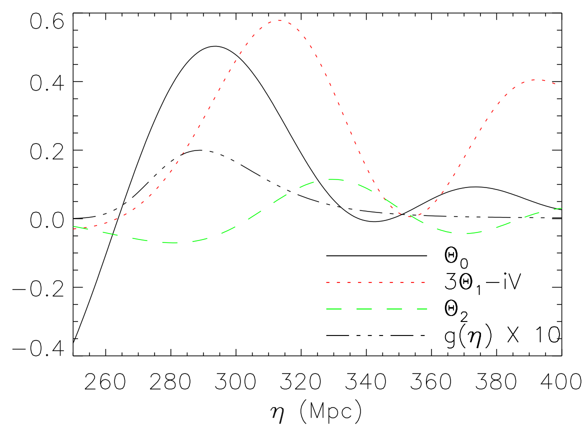

FIG. 5. $\Theta_0^{(1)}$, $3\Theta_1^{(1)} - iv^{(1)}$ and $\Theta_2^{(1)}$ as a function of η for wavenumber $k = 0.1\,\mathrm{Mpc}^{-1}$. Note that at small scales $3\Theta_1^{(1)} - iv^{(1)}$ becomes comparable to $\Theta_0^{(1)}$, but its contribution to the bispectrum is suppressed because it is weighted by the derivative of spherical Bessel function. See also Equation 10 and Figure 3 in KW09.

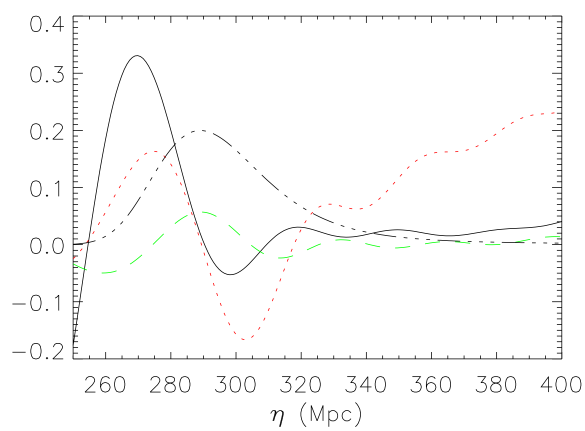

FIG. 6. $\Theta_0^{(1)}$, $3\Theta_1^{(1)} - iv^{(1)}$ and $\Theta_2^{(1)}$ as a function of η for wavenumber $k = 0.2\,\mathrm{Mpc}^{-1}$. Note that at small scales $3\Theta_1^{(1)} - iv^{(1)}$ becomes comparable to $\Theta_0^{(1)}$, but its contribution to the bispectrum is suppressed because it is weighted by the derivative of spherical Bessel function. See also Equation 10 and Figure 3 in KW09.

than or at most of similar magnitude as the last term in Equation 34. As we will see later the last term in Equation 34 gives only $\sim 5\%$ contribution to signal to noise and is thus not important.

Before presenting the numerical results we note that $S_{\delta v}$ remains small until the very

end of recombination. By the time $S_{\delta v}$ finally becomes somewhat larger the visibility function becomes small suppressing the contribution to the CMB anisotropies. Figures 3,4, 5 and 6 show comparison between $\Theta_0^{(1)}$, $3\Theta_1^{(1)} - iv^{(1)}$ and $\Theta_2^{(1)}$ for wavenumbers $k = 0.001\,\mathrm{Mpc}^{-1}, 0.01\,\mathrm{Mpc}^{-1}$, $0.1\,\mathrm{Mpc}^{-1}$ and $0.2\,\mathrm{Mpc}^{-1}$. In interpreting these figures it should be kept in mind that $3\Theta_1^{(1)} - iv^{(1)}$ is weighted by the derivative of the spherical Bessel function (Equation 10 in KW09) in the expression for bispectrum which is smaller than the spherical Bessel function by about an order of magnitude near the peak. Thus even though in Figure 5 and 6 $3\Theta_1^{(1)} - iv^{(1)}$ seems comparable in magnitude to $\Theta_0^{(1)}$ its contribution to the bispectrum is much smaller.

We will collectively refer to the source terms calculated in KW09 as S^{KW09}, that is all the terms on the right hand side of Equation 9 except $\Theta_{00}^{(2)}, V^{(2)}$ and $\Theta_{2m}^{(2)}$. Figure 7 shows the confusion with primordial bispectrum of local type as parameterized by f_{NL} defined in KW09 as a function of maximum ℓ mode measured by an ideal experiment due to $\Theta_{2m}^{(2)} = 10/9 S_{\delta 2}^m$ and $V_m^{(2)} = -1/(1+R) S_{\delta v}^m$. For $\ell_{max} = 2500$ we get $f_{NL} \sim -0.02$, a few percent of the value found in KW09 for S^{KW09}. An important point to note is that the sign of the bispectrum at small scales is same as the net contribution from S^{KW09}. Thus the new terms calculated here will add to the bispectrum from S^{KW09} and should increase S/N by a small amount.

In Figure 8 we show the signal to noise ratio for the detection of the bispectrum generated by inhomogeneous recombination for a cosmic variance limited experiment as a function of the maximum multipole moment observed ℓ_{max} [29]

$$\begin{aligned} \frac{S}{N} &\equiv \frac{1}{\sqrt{F_{rec}^{-1}}}, \\ F_{rec} &= \sum_{\ell_1 \le \ell_2 \le \ell_3 \le \ell_{\max}} \frac{(B_{rec}^{\ell_1 \ell_2 \ell_3})^2}{\Delta_{\ell_1 \ell_2 \ell_3} C_{\ell_1} C_{\ell_2} C_{\ell_3}}, \\ \Delta_{\ell_1 \ell_2 \ell_3} &\equiv 1 + \delta_{\ell_1 \ell_2} + \delta_{\ell_2 \ell_3} + \delta_{\ell_3 \ell_1} + 2\delta_{\ell_1 \ell_2}\delta_{\ell_2 \ell_3}, \end{aligned} \tag{36}$$

where $B_{rec}^{\ell_1 \ell_2 \ell_3}$ is the angular averaged bispectrum generated by inhomogeneous recombination , C_ℓ is the CMB angular power spectrum and $\delta_{\ell_1 \ell_2}$ is the Kronecker delta function. We get $S/N \sim 1$ at $\ell_{max} = 2500$. Contributions from S^{KW09} and $\Theta_{2m}^{(2)}$ and $V_m^{(2)}$ calculated in this paper are shown separately. S^{KW09} give $S/N \sim 1$ compared with $S/N \sim 0.05$ contributed by the second order baryon velocity and second order quadrupole. A future high resolution cosmic variance limited experiment may thus see a hint of inhomogeneous recombination in the bispectrum.

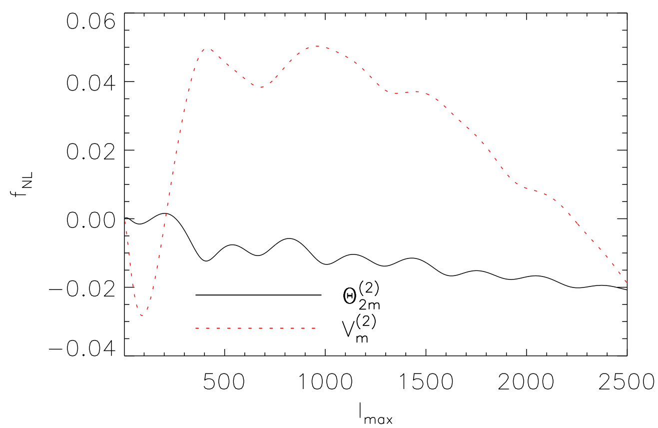

FIG. 7. Confusion with primordial non-Gaussianity parameterized by f_{NL}. Contribution of $\Theta^{(2)}_{2m} = 10/9 S^m_{\delta 2}$ and $V^{(2)}_m = -1/(1+R) S^m_{\delta v}$ is only a few per cent of the contribution from S^{KW09}, the source terms calculated in KW09. S^{KW09} gives a cumulative contribution of $f_{NL} \sim -1$ at $\ell_{max} = 2500$. The calculations were done including Fourier modes up to $k = 0.5\,\mathrm{Mpc}^{-1}$. Contributions from $k \gtrsim 0.4\,\mathrm{Mpc}^{-1}$ are negligible.

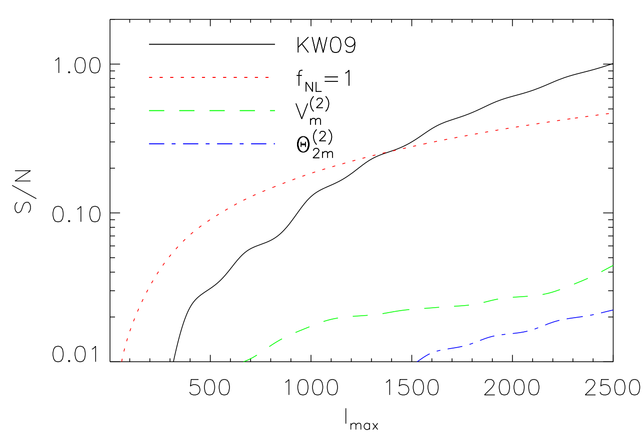

FIG. 8. Signal to noise ratio for the bispectrum generated by inhomogeneous recombination for a cosmic variance limited experiment as a function of the maximum multipole moment ℓ_{max}. S/N due to S^{KW09} is ~ 1 for $\ell_{max} = 2500$. Contribution due to $\Theta^{(2)}_{2m} = 10/9 S^m_{\delta 2}$ and $V^{(2)}_m = -1/(1+R) S^m_{\delta v}$ is only a few percent of the contributions S^{KW09}. Also shown for comparison is S/N from primordial non-Gaussianity with $f_{NL} = 1$. The calculations were done including Fourier modes up to $k = 0.5\,\mathrm{Mpc}^{-1}$. Contributions from $k \gtrsim 0.4\,\mathrm{Mpc}^{-1}$ are negligible.

VII. CONCLUSIONS

We have analyzed two different ways of integrating the second order photon Boltzmann equations. It is necessary for the consistency of perturbation theory that it should not matter if you solve different perturbation orders together or separately and we find that it is so in this case. We can define a typical second order term to be of the form $\Theta_0^{(1)} \times \Theta_0^{(1)}$ with a prefactor of order unity and which can be expected to give rise to a local type non-Gaussianity parameter $|f_{NL}| \sim 1$. Then we have shown that the second order monopole, dipole and quadrupole are smaller than typical second order terms. Although we have derived this result in the tight coupling limit to second order in $R/\dot{\tau}$, the fact that these terms are small is valid in general. This is because the cancellation that causes these terms to be small occurs in the original Boltzmann equations.

It can be seen that perturbing the electron number density in the first order monopole, dipole and quadrupole solutions does not work as follows. The full first order solution can be approximately written as a product of an oscillating part and a damping part. Senatore et al. [23] perturb just the damping part to estimate the second order solution. The oscillating part of the solution does not contain explicit dependence on the electron number density but the equations used in arriving at that solution do depend on the electron number density [26]. To get the oscillating part we have to expand the baryon momentum equation to first order in $R/\dot{\tau}$. The factor of $\dot{\tau}$ however cancels when the baryon momentum equation is substituted into the photon Boltzmann equation and does not explicitly show up in the resulting oscillating solution. Similar cancellation happens for the damping solution as well. When the electron number density is perturbed in the original equations these additional factors of $\dot{\tau}$ lead to additional terms in the second order equation that depend on electron number density perturbation. Thus there is no way to perturb the electron number density in the first order oscillating and damping solutions to take into account these extra second order terms and the only way to get the correct second order solution is to solve the second order Boltzmann hierarchy explicitly as we have done. In particular the terms missed come from the $\delta_e\Theta^{(1)}$ term in the second order Boltzmann equation which also results in the cancellation of the first order monopole in the second order Boltzmann hierarchy and gives the second δ_e term in Equation 8.

In addition the correct solution should satisfy the relation between the second order

monopole and dipole, Equation 17 (first equation in the Boltzmann hierarchy). The solutions given in Senatore et al. [23] clearly fail to satisfy this relation. In particular this relation says that the second order monopole and dipole should have the same dependence on angular wavenumbers, the factors of $Y_{\ell m}(\hat{\mathbf{k}})$. The first order solutions are the solutions for the transfer functions and depend on only the wavenumber magnitude. So it is not surprising that perturbing the first order solutions fails to capture the angular dependence of the second order solutions.

Physically what the absence of the first order monopole from the second order Boltzmann equations means is that if we have a uniform radiation field then scattering by a stationary inhomogeneous distribution of electrons does not introduce additional inhomogeneities in the radiation field (in the elastic Thomson scattering limit). The dipole seen in the electron rest frame contributes to the additional inhomogeneities in the radiation field but it is small during recombination. Our analysis justifies neglecting the second order monopole, dipole and quadrupole, as we did in KW09. In particular, we conclude, as in KW09, the confusion with the primordial non-Gaussianity of local type resulting from inhomogeneous recombination is $|f_{NL}| \lesssim 1$ and thus not important for the Planck satellite mission [3] which is predicted to achieve an accuracy of $\Delta f_{NL} \sim 5$ [30, 31]. The S/N for the detection of this bispectrum by an ideal full sky experiment using temperature data alone is ~ 1. However perturbations in the electron number density will also have an effect on CMB polarization. If this effect is of a magnitude comparable or larger than the effect on temperature, a post-Planck, high-resolution, all-sky mission measuring the CMB temperature and polarization anisotropies may see the imprint of inhomogeneous recombination in the CMB bispectrum at few sigma level.

ACKNOWLEDGMENTS

We thank Leonardo Senatore, Svetlin Tassev, and Matias Zaldarriaga for comments on the manuscript. This work was supported by Campus Research Board, University of Illinois and NSF grant AST 07-08849. BDW acknowledges the Galileo Galilei Institute for hospitality.

[1] B. Novosyadlyj, Mon. Not. R. Astron. Soc. **370**, 1771 (Aug. 2006).

[2] L. Senatore, S. Tassev, and M. Zaldarriaga, ArXiv e-prints(Dec. 2008), arXiv:0812.3652.

[3] Planck Science Team, "Planck blue book," (2005), `http://www.rssd.esa.int/Planck`.

[4] W. Hu, D. Scott, and J. Silk, Phys. Rev. D **49**, 648 (Jan. 1994), arXiv:astro-ph/9305038.

[5] S. Dodelson and J. M. Jubas, Astrophys. J. **439**, 503 (Feb. 1995), arXiv:astro-ph/9308019.

[6] T. Pyne and S. M. Carroll, Phys. Rev. D **53**, 2920 (Mar. 1996), arXiv:astro-ph/9510041.

[7] S. Mollerach and S. Matarrese, Phys. Rev. D **56**, 4494 (Oct. 1997), arXiv:astro-ph/9702234.

[8] S. Matarrese, S. Mollerach, and M. Bruni, Phys. Rev. D **58**, 043504 (Aug. 1998), arXiv:astro-ph/9707278.

[9] D. M. Goldberg and D. N. Spergel, Phys. Rev. D **59**, 103002 (May 1999), arXiv:astro-ph/9811251.

[10] P. Creminelli and M. Zaldarriaga, Phys. Rev. D **70**, 083532 (Oct. 2004), arXiv:astro-ph/0405428.

[11] N. Bartolo, S. Matarrese, and A. Riotto, Physical Review Letters **93**, 231301 (Dec. 2004), arXiv:astro-ph/0407505.

[12] K. Tomita, Phys. Rev. D **71**, 083504 (Apr. 2005), arXiv:astro-ph/0501663.

[13] N. Bartolo, S. Matarrese, and A. Riotto, Journal of Cosmology and Astro-Particle Physics **8**, 10 (Aug. 2005), arXiv:astro-ph/0506410.

[14] N. Bartolo, S. Matarrese, and A. Riotto, Journal of Cosmology and Astro-Particle Physics **5**, 10 (May 2006), arXiv:astro-ph/0512481.

[15] K. Tomita, Phys. Rev. D **77**, 103521 (May 2008), arXiv:0712.2511.

[16] C. Pitrou, J.-P. Uzan, and F. Bernardeau, Phys. Rev. D **78**, 063526 (Sep. 2008), arXiv:0807.0341.

[17] N. Bartolo, S. Matarrese, and A. Riotto, Journal of Cosmology and Astro-Particle Physics **6**, 24 (Jun. 2006), arXiv:astro-ph/0604416.

[18] N. Bartolo, S. Matarrese, and A. Riotto, Journal of Cosmology and Astro-Particle Physics **1**, 19 (Jan. 2007), arXiv:astro-ph/0610110.

[19] N. Bartolo and A. Riotto, ArXiv e-prints(Nov. 2008), arXiv:0811.4584.

[20] C. Pitrou, Classical and Quantum Gravity **24**, 6127 (Dec. 2007), arXiv:0706.4383.

[21] C. Pitrou, ArXiv e-prints(Sep. 2008), arXiv:0809.3036.

[22] R. Khatri and B. D. Wandelt, Phys. Rev. D **79**, 023501 (Jan. 2009), arXiv:0810.4370.

[23] L. Senatore, S. Tassev, and M. Zaldarriaga, ArXiv e-prints(Dec. 2008), arXiv:0812.3658.

[24] U. Seljak and M. Zaldarriaga, Astrophys. J. **469**, 437 (Oct. 1996).

[25] N. Bartolo, S. Matarrese, and A. Riotto, ArXiv Astrophysics e-prints(Mar. 2007), arXiv:astro-ph/0703496.

[26] S. Dodelson, *Modern cosmology* (Modern cosmology / Scott Dodelson. Amsterdam (Netherlands): Academic Press. ISBN 0-12-219141-2, 2003).

[27] In [25] $\delta_e^{(1)}$ is assumed to be equal to $\delta_b^{(1)}$ in writing the momentum equation for baryons, [2] give the momentum equation for baryons without this assumption.

[28] W. Hu and N. Sugiyama, Astrophys. J. **444**, 489 (May 1995), arXiv:astro-ph/9407093.

[29] E. Komatsu and D. N. Spergel, Phys. Rev. D **63**, 063002 (Mar. 2001), arXiv:astro-ph/0005036.

[30] D. Babich and M. Zaldarriaga, Phys. Rev. D **70**, 083005 (Oct. 2004), arXiv:astro-ph/0408455.

[31] A. P. S. Yadav, E. Komatsu, and B. D. Wandelt, Astrophys. J. **664**, 680 (Aug. 2007), arXiv:astro-ph/0701921.

[32] D. A. Varshalovich, A. N. Moskalev and V. K. Khersonskii, *Quantum theory of angular momentum* (World scientific, Singapore, 1988).

[33] S. Weinberg, Phys. Rev. D **74**, 063517 (Sep. 2006), arXiv:astro-ph/0607076.

[34] D. Baskaran, L. P. Grishchuk, and A. G. Polnarev, Phys. Rev. D **74**, 083008 (Oct. 2006), arXiv:gr-qc/0605100.

Appendix A: Contribution from $\Theta_{00}^{(2)}$ and $V_m^{(2)}$

We can write the formal solution for $\Theta^{(2)}(\mathbf{k}, \hat{\mathbf{n}}, \eta_0)$,

$$\Theta^{(2)}(\mathbf{k}, \hat{\mathbf{n}}, \eta_0) = \int_0^{\eta_0} d\eta e^{ik(\eta-\eta_0)\hat{\mathbf{k}}.\hat{\mathbf{n}}} e^{-\tau} S^{(2)}(\mathbf{k}, \hat{\mathbf{n}}, \eta). \tag{A1}$$

We will first include only the first term in Equation 29 in the source $S^{(2)}(\mathbf{k}, \hat{\mathbf{n}}, \eta)$. The calculation for other terms is similar.

The angular averaged bispectrum is defined as sum over the $m's$ of bispectrum times a Wigner 3jm symbol.

$$\begin{aligned} B^{\ell_1\ell_2\ell_3} &= \sum_{m_1 m_2 m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B^{\ell_1\ell_2\ell_3}_{m_1 m_2 m_3} \\ &= \sum_{m_1 m_2 m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \langle a^{(1)}_{\ell_1 m_1}(\boldsymbol{x}, \eta_0) a^{(1)}_{\ell_2 m_2}(\boldsymbol{x}, \eta_0) a^{(2)}_{\ell_3 m_3}(\boldsymbol{x}, \eta_0)\rangle + \text{ 2 permutations} \end{aligned} \tag{A2}$$

where $a^{(2)}_{\ell m}$ is the Fourier transform of $\Theta^{(2)}_{\ell m}$ and $a^{(1)}_{\ell m}$ is calculated from first order multipole moments $\Theta^{(1)}_{\ell}$.

$$a^{(2)}_{\ell m}(\boldsymbol{x},\eta_0) = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}.\boldsymbol{x}} \Theta^{(2)}_{\ell m}(\boldsymbol{k},\eta_0)$$
$$a^{(1)}_{\ell m}(\boldsymbol{x},\eta_0) = 4\pi \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} e^{i\boldsymbol{k}.\boldsymbol{x}} (-i)^{\ell} \Theta^{(1)}_{\ell}(\boldsymbol{k},\eta_0) Y^*_{\ell m}(\hat{\boldsymbol{k}})$$

Proceeding as in KW09 we get for the bispectrum from the first term in Equation 29

$$\begin{aligned} B^{\ell_1\ell_2\ell_3}_{m_1m_2m_3} = &-(4\pi)^2(2\pi)^3 \int_0^{\eta_0} d\eta g(\eta) \int \frac{d^3\boldsymbol{k_1}}{(2\pi)^3}\frac{d^3\boldsymbol{k_2}}{(2\pi)^3}\frac{d^3\boldsymbol{k_3}}{(2\pi)^3} (-i)^{\ell_1+\ell_2+\ell_3} Y^*_{l_1m_1}(\hat{\boldsymbol{k_1}})Y^*_{\ell_2m_2}(\hat{\boldsymbol{k_2}}) \\ & P(k_1)P(k_2)(4\pi)^{3/2}\int_0^{\eta} d\eta' \frac{4\sqrt{5}}{9} k_3 c_s(\eta') \sin\left[k_3(r_s(\eta)-r_s(\eta'))\right] j_{\ell_3}\left[k_3(\eta-\eta_0)\right] \\ & Y^*_{\ell_3m_3}(\hat{\mathbf{k_3}})Y^*_{20}(-\hat{\mathbf{k_2}})\delta_e(k_1,\eta')\Theta^{(1)}_2(k_2,\eta')\Theta^{(1)}_{\ell_1}(k_1,\eta_0)\Theta^{(1)}_{\ell_2}(k_2,\eta_0)\delta^3(\boldsymbol{k_1}+\boldsymbol{k_2}+\boldsymbol{k_3}) \\ & + 5 \text{ permutations} \end{aligned} \quad \text{(A3)}$$

We have ignored $\Pi^{(1)}$ in $S^{(0)}_{\delta 2}$ to simplify equations, including it at the end of the calculation is trivial. We now use the Dirac delta distribution to integrate over $\mathbf{k_3}$.

$$\begin{aligned} B^{\ell_1\ell_2\ell_3}_{m_1m_2m_3} = &-(4\pi)^2 \int_0^{\eta_0} d\eta g(\eta) \int \frac{d^3\boldsymbol{k_1}}{(2\pi)^3}\frac{d^3\boldsymbol{k_2}}{(2\pi)^3} (-i)^{\ell_1+\ell_2} i^{\ell_3} P(k_1)P(k_2)\Theta^{(1)}_{\ell_1}(k_1,\eta_0)\Theta^{(1)}_{\ell_2}(k_2,\eta_0) \\ & (4\pi)^{3/2}\int_0^{\eta} d\eta' \frac{4\sqrt{5}}{9} |\mathbf{k_1}+\mathbf{k_2}| c_s(\eta') \sin\left[|\mathbf{k_1}+\mathbf{k_2}|(r_s(\eta)-r_s(\eta'))\right] j_{\ell_3}\left[|\mathbf{k_1}+\mathbf{k_2}|(\eta-\eta_0)\right] \\ & Y^*_{l_1m_1}(\hat{\boldsymbol{k_1}})Y^*_{\ell_2m_2}(\hat{\boldsymbol{k_2}})Y^*_{\ell_3m_3}(-(\widehat{\mathbf{k}_1+\mathbf{k}_2}))Y^*_{20}(-\hat{\mathbf{k_2}})\delta_e(k_1,\eta')\Theta^{(1)}_2(k_2,\eta') \\ & + 5 \text{ permutations} \end{aligned} \quad \text{(A4)}$$

To proceed further we will need the following addition theorem for spherical waves [32]

$$\begin{aligned} z_L(|\mathbf{k_1}+\mathbf{k_2}|r)Y_{LM}(\widehat{\mathbf{k}_1+\mathbf{k}_2}) = &\sum_{\ell_1\ell_2m_1m_2} i^{\ell_1+\ell_2-L}(-1)^M\sqrt{4\pi(2L+1)(2\ell_1+1)(2\ell_2+1)} j_{\ell_1}(k_1r)z_{\ell_2}(k_2r) \\ & \begin{pmatrix} \ell_1 & \ell_2 & L \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1 & m_2 & -M \end{pmatrix} Y_{\ell_1m_1}(\hat{\boldsymbol{k_1}})Y_{\ell_2m_2}(\hat{\boldsymbol{k_2}}), \end{aligned} \quad \text{(A5)}$$

where z_ℓ is any of the spherical Bessel function and the sum is over all allowed values of ℓ_1,ℓ_2,m_1,m_2. The above equation is valid for arbitrary values of k_1 and k_2 if $z_\ell = j_\ell$, the spherical Bessel function of first kind. If $z_\ell = y_\ell$, the spherical Bessel function of second kind, then Equation A5 is valid for $k_1 < k_2$ (and for $k_2 < k_1$ after interchanging k_1 and k_2).

We now use A5 for the product $j_{\ell_3}Y^*_{\ell_3m_3}$. We also write $\sin\left[|\mathbf{k_1}+\mathbf{k_2}|(r_s(\eta)-r_s(\eta'))\right] = \left[|\mathbf{k_1}+\mathbf{k_2}|(r_s(\eta)-r_s(\eta'))\right] j_0\left[|\mathbf{k_1}+\mathbf{k_2}|(r_s(\eta)-r_s(\eta'))\right]$ and use Equation A5 again. We also

use

$$|\mathbf{k_1}+\mathbf{k_2}|^2 = k_1^2+k_2^2+\frac{8\pi}{3}k_1k_2\sum_{m'}Y^*_{1m'}(\hat{\mathbf{k}}_\mathbf{1})Y_{1m'}(\hat{\mathbf{k}}_\mathbf{2}) \tag{A6}$$

The angular integrals over $\hat{\mathbf{k}}_\mathbf{1}$ and $\hat{\mathbf{k}}_\mathbf{2}$ can now be done. Right hand side of Equation A6 consists of two terms: $k_1^2+k_2^2$ has no angular dependence while the rest of the right hand side depends on the angles $\hat{\mathbf{k}}_\mathbf{1}$ and $\hat{\mathbf{k}}_\mathbf{2}$. For simplicity we will show the calculation for only $k_1^2+k_2^2$ part. The calculation for the other part is similar but since we have extra factors of spherical harmonics we will get extra Wigner 3jm symbols on integration over angles summing over which will require few extra steps.

The result for $k_1^2+k_2^2$ part is

$$\begin{aligned}
&-\frac{(4\pi)^3}{(2\pi)^6}\int_0^{\eta_0}d\eta g(\eta)\int dk_1k_1^2\int dk_2k_2^2(-i)^{\ell_1+\ell_2}P(k_1)P(k_2)\Theta^{(1)}_{\ell_1}(k_1,\eta_0)\Theta^{(1)}_{\ell_2}(k_2,\eta_0)\\
&\int_0^{\eta}d\eta'\frac{4\sqrt{5}}{9}\left(k_1^2+k_2^2\right)\left(r_s(\eta)-r_s(\eta')\right)c_s(\eta')\delta_e(k_1,\eta')\Theta^{(1)}_2(k_2,\eta')\sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}}\\
&\sum_{\ell''\ell_1'\ell_2'Lm''m_1'm_2'M}(-1)^{\ell''+\ell_3+m_1}i^{\ell_1'+\ell_2'}(2\ell_1'+1)(2\ell_2'+1)(2\ell''+1)(2L+1)\\
&j_{\ell_1'}\left[(\eta-\eta_0)k_1\right]j_{\ell_2'}\left[(\eta-\eta_0)k_2\right]j_{\ell''}\left[(r_s(\eta)-r_s(\eta'))k_1\right]j_{\ell''}\left[(r_s(\eta)-r_s(\eta'))k_2\right]\\
&\begin{pmatrix}\ell_1' & \ell_2' & \ell_3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\ell'' & \ell_1' & \ell_1\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}2 & \ell_2' & L\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\ell_2 & \ell'' & L\\ 0 & 0 & 0\end{pmatrix}\\
&\begin{pmatrix}\ell_1' & \ell_2' & \ell_3\\ m_1' & m_2' & m_3\end{pmatrix}\begin{pmatrix}\ell'' & \ell_1' & \ell_1\\ m'' & m_1' & -m_1\end{pmatrix}\begin{pmatrix}2 & \ell_2' & L\\ 0 & m_2' & -M\end{pmatrix}\begin{pmatrix}\ell_2 & \ell'' & L\\ m_2 & m'' & -M\end{pmatrix}
\end{aligned} \tag{A7}$$

Summing over the $m's$ we get [32]

$$\begin{aligned}
&\sum_{m''m_1'm_2'M}(-1)^{\ell''+\ell_3+m_1}\begin{pmatrix}\ell_1' & \ell_2' & \ell_3\\ m_1' & m_2' & m_3\end{pmatrix}\begin{pmatrix}\ell'' & \ell_1' & \ell_1\\ m'' & m_1' & -m_1\end{pmatrix}\begin{pmatrix}2 & \ell_2' & L\\ 0 & m_2' & -M\end{pmatrix}\begin{pmatrix}\ell_2 & \ell'' & L\\ m_2 & m'' & -M\end{pmatrix}\\
&=\sum_{L'M'}(-1)^{L+\ell''+\ell_2'+\ell_3+\ell_1+L'-m_1-m_2-m_3-M'}(2L'+1)\begin{pmatrix}\ell_3 & L' & \ell_1\\ m_3 & -M' & m_1\end{pmatrix}\begin{pmatrix}2 & L' & \ell_2\\ 0 & M' & m_2\end{pmatrix}\\
&\begin{Bmatrix}\ell_3 & L' & \ell_1\\ \ell'' & \ell_1' & \ell_2'\end{Bmatrix}\begin{Bmatrix}2 & L' & \ell_2\\ \ell'' & L & \ell_2'\end{Bmatrix},
\end{aligned} \tag{A8}$$

where the matrices in the last line are the $6j$ symbols. All the m dependence of the bispectrum is in the above expression. Therefore to calculate the angular averaged bispectrum we need only consider the above expression for averaging over m_1, m_2, m_3. The result of doing

this averaging is

$$
\begin{aligned}
&\sum_{L'M'm_1m_2m_3}(-1)^{L+\ell''+\ell_2'+\ell_3+\ell_1+L'-m_1-m_2-m_3-M'}(2L'+1)\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}\\
&\begin{pmatrix}\ell_3 & L' & \ell_1\\ m_3 & -M' & m_1\end{pmatrix}\begin{pmatrix}2 & L' & \ell_2\\ 0 & M' & m_2\end{pmatrix}\begin{Bmatrix}\ell_3 & L' & \ell_1\\ \ell'' & \ell_1' & \ell_2'\end{Bmatrix}\begin{Bmatrix}2 & L' & \ell_2\\ \ell'' & L & \ell_2'\end{Bmatrix}\\
=&\sum_{L'm_3}(-1)^{L+\ell''+\ell_2'+\ell_1+\ell_2+L'-m_3}(2L'+1)\begin{pmatrix}\ell_3 & \ell_3 & 2\\ -m_3 & m_3 & 0\end{pmatrix}\\
&\begin{Bmatrix}\ell_3 & \ell_3 & 2\\ L' & \ell_2 & \ell_1\end{Bmatrix}\begin{Bmatrix}\ell_3 & L' & \ell_1\\ \ell'' & \ell_1' & \ell_2'\end{Bmatrix}\begin{Bmatrix}2 & L' & \ell_2\\ \ell'' & L & \ell_2'\end{Bmatrix}\\
=&\sum_{L'}(-1)^{L+\ell''+\ell_2'+\ell_1+\ell_2+\ell_3+L'}(2L'+1)\sqrt{(2\ell_3+1)}\delta_{20}\delta_{00}\\
&\begin{Bmatrix}\ell_3 & \ell_3 & 2\\ L' & \ell_2 & \ell_1\end{Bmatrix}\begin{Bmatrix}\ell_3 & L' & \ell_1\\ \ell'' & \ell_1' & \ell_2'\end{Bmatrix}\begin{Bmatrix}2 & L' & \ell_2\\ \ell'' & L & \ell_2'\end{Bmatrix}\\
=&\,0
\end{aligned}
\tag{A9}
$$

The calculation for the other term in A6 is similar and it also results in the Kronecker delta symbol $\delta_{20} = 0$.

The second term in Equation 29 involves cosine which can be written in terms of the spherical Bessel function of the second kind, y_0. We therefore need to break the integral over (k_1, k_2) in two parts , $k_1 > k_2$ and $k_1 < k_2$ in order to apply the addition theorem. Both the terms will give a zero contribution to the angular averaged bispectrum (with δ_{10} in the final result due to Y_{10} in this term), which is easily shown by a calculation similar to above. The boundary $k_1 = k_2$ will also give zero contribution to the (k_1, k_2) integral because the integrand is finite.

Thus we have shown that the contribution from $\Theta^{(2)}_{00}$ to the angular averaged bispectrum vanishes. A similar calculation for the $V^{(2)}_m$ shows that the contribution from the terms involving $S_{\delta 2}$ in Equation 34 also gives zero contribution to the angular averaged bispectrum. In general $\Theta^{(2)}_{LM} \sim \delta_e \Theta^{(1)}_\ell Y_{\ell m}$ gives non-zero contribution to the angular averaged bispectrum if and only if $L = \ell$ and $M = m$ because of the orthogonality of spherical harmonics of different orders.

Appendix B: Integral Equation for second order monopole

An alternative to solving the Boltzmann hierarchy for the second order monopole is to solve an integral equation [33, 34]. The line of sight solution for second order Boltzmann equation is

$$\Theta^{(2)}(\eta,\mathbf{k},\hat{\mathbf{n}}) = e^{\tau(\eta)}\int_0^{\eta} d\eta' e^{i\mathbf{k}.\hat{\mathbf{n}}(\eta'-\eta)} g(\eta')\left[\int \frac{d^3k'}{(2\pi)^3}\delta_e^{(1)}(\mathbf{k}')4\pi\sum_{\ell''m''}(-i)^{\ell''}f_{\ell''}(\mathbf{k}-\mathbf{k}',\eta')Y_{\ell''m''}(\hat{\mathbf{n}})\right.$$
$$\left. Y^*_{\ell''m''}(\widehat{\mathbf{k}-\mathbf{k}'}) + \frac{1}{\sqrt{4\pi}}\Theta^{(2)}_{00}(\mathbf{k},\eta') + \frac{1}{10}\sum_{m''}\Theta^{(2)}_{2m''}(\mathbf{k},\eta')Y_{2m''}(\hat{\mathbf{n}}) + \sum_{m''}v^{(2)}_{m''}(\mathbf{k},\eta')Y_{1m''}(\hat{\mathbf{n}})\right], \tag{B1}$$

where f_ℓ represents a general first order term multiplying δ_e. We can integrate over direction $\hat{\mathbf{n}}$ to get an integral equation for the monopole

$$\Theta^{(2)}_{00}(\eta,\mathbf{k}) = e^{\tau(\eta)}\int_0^{\eta} d\eta' g(\eta')\left[(4\pi)^{3/2}\int \frac{d^3k'}{(2\pi)^3}\delta_e^{(1)}(\mathbf{k}')\sum_{\ell''m''}j_{\ell''}\left[k(\eta'-\eta)\right]f_{\ell''}(\mathbf{k}-\mathbf{k}',\eta')\right.$$
$$Y_{\ell''m''}(\hat{\mathbf{k}})Y^*_{\ell''m''}(\widehat{\mathbf{k}-\mathbf{k}'}) + j_0\left[k(\eta'-\eta)\right]\Theta^{(2)}_{00}(\mathbf{k},\eta') - \frac{\sqrt{4\pi}}{10}j_2\left[k(\eta'-\eta)\right]$$
$$\left.\sum_{m''}\Theta^{(2)}_{2m''}(\mathbf{k},\eta')Y_{2m''}(\hat{\mathbf{k}}) + i\sqrt{4\pi}j_1\left[k(\eta'-\eta)\right]\sum_{m''}v^{(2)}_{m''}(\mathbf{k},\eta')Y_{1m''}(\hat{\mathbf{k}})\right], \tag{B2}$$

We can now write down the contribution of $\Theta^{(2)}_{00}$ to the bispectrum

$$B^{m_1m_2m_3}_{\ell_1\ell_2\ell_3} = \int_0^{\eta_0} d\eta g(\eta) S^{m_1m_2m_3}_{\ell_1\ell_2\ell_3}(\eta) + 2\text{ permutations},$$
$$S^{m_1m_2m_3}_{\ell_1\ell_2\ell_3}(\eta) \equiv (4\pi)^3\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3k_3}{(2\pi)^3}(-i)^{\ell_1+\ell_2}i^{\ell_3}Y^*_{\ell_1m_1}(\hat{\mathbf{k_1}})Y^*_{\ell_2m_2}(\hat{\mathbf{k_2}})Y^*_{\ell_3m_3}(\hat{\mathbf{k_3}})j_{\ell_3}\left[k_3(\eta-\eta_0)\right]$$
$$\langle\frac{1}{\sqrt{4\pi}}\Theta_{00}(\mathbf{k_3},\eta)\Theta_{\ell_1}(\mathbf{k_1},\eta_0)\Theta_{\ell_2}(\mathbf{k_2},\eta_0)\rangle$$
$$= (4\pi)^3\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3k_3}{(2\pi)^3}(-i)^{\ell_1+\ell_2}i^{\ell_3}Y^*_{\ell_1m_1}(\hat{\mathbf{k_1}})Y^*_{\ell_2m_2}(\hat{\mathbf{k_2}})Y^*_{\ell_3m_3}(\hat{\mathbf{k_3}})j_{\ell_3}\left[k_3(\eta-\eta_0)\right]$$
$$e^{\tau(\eta)}\int_0^{\eta}d\eta' g(\eta')\left[4\pi\int\frac{d^3k'}{(2\pi)^3}\sum_{\ell''m''}j_{\ell''}\left[k_3(\eta'-\eta)\right]Y_{\ell''m''}(\hat{\mathbf{k_3}})Y^*_{\ell''m''}(\widehat{\mathbf{k_3}-\mathbf{k}'})\right]$$
$$\langle\delta_e^{(1)}(\mathbf{k}')f_{\ell''}(\mathbf{k_3}-\mathbf{k}',\eta')\Theta_{\ell_1}(\mathbf{k_1},\eta_0)\Theta_{\ell_2}(\mathbf{k_2},\eta_0)\rangle$$
$$+\frac{1}{\sqrt{4\pi}}j_0\left[k_3(\eta'-\eta)\right]\langle\Theta^{(2)}_{00}(\mathbf{k_3},\eta')\Theta_{\ell_1}(\mathbf{k_1},\eta_0)\Theta_{\ell_2}(\mathbf{k_2},\eta_0)\rangle$$
$$+ij_1\left[k_3(\eta'-\eta)\right]\sum_{m''}Y_{1m''}(\hat{\mathbf{k}}_3)\langle v^{(2)}_{m''}(\mathbf{k_3},\eta')\Theta_{\ell_1}(\mathbf{k_1},\eta_0)\Theta_{\ell_2}(\mathbf{k_2},\eta_0)\rangle$$
$$\left.-\frac{1}{10}j_2\left[k_3(\eta'-\eta)\right]\sum_{m''}Y_{2m''}(\hat{\mathbf{k}}_3)\langle\Theta^{(2)}_{2m''}(\mathbf{k_3},\eta')\Theta_{\ell_1}(\mathbf{k_1},\eta_0)\Theta_{\ell_2}(\mathbf{k_2},\eta_0)\rangle\right] \tag{B3}$$

Here we have used the integral equation for $\Theta_{00}^{(2)}$ (Equation B2) to get an equation for $S_{\ell_1\ell_2\ell_3}^{m_1m_2m_3}$. The last term involving $\Theta_{2m''}^{(2)}$ will give a small contribution ($\sim 10\%$) because of the factor of 1/10 and can be neglected. For $v_m^{(2)}$ we can use the approximate tight coupling solution, the last term in Equation 34, in which case it can be absorbed into $f_{\ell''}$ for $\ell''=1$. We can similarly absorb the last term also if we choose not to neglect it. If we did not have a factor of j_0 multiplying the second order monopole term in last but third line, we would have an integral equation for $S_{\ell_1\ell_2\ell_3}^{m_1m_2m_3}$. We can however make progress by using the approximate solution for the second order monopole Equation 29. Then a calculation similar to Appendix A shows that the contribution of this term to the reduced bispectrum is exactly zero, so this term can be dropped. For the other terms we proceed as in KW09 and Appendix A. We break the four point correlation function of first order terms into two point correlation functions using Wick's theorem. We can then perform all the angular integrals and two of the radial integrals using the properties of Dirac delta distribution, spherical harmonics and Wigner $3jm$ and $6j$ symbols. The result is

$$
\begin{aligned}
S_{\ell_1\ell_2\ell_3}^{m_1m_2m_3}(\eta) = {} & \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \frac{(4\pi)^4}{(2\pi)^6} \int dk_1 k_1^2 \int dk_2 k_2^2 \\
& e^{\tau(\eta)} \int_0^\eta d\eta' g(\eta') \sum_{\ell''\ell_1'\ell_2'\ell_1''\ell_2''} f_{\ell''}(k_1,\eta')\delta_e(k_2,\eta')\Theta_{\ell_1}(k_1,\eta_0)\Theta_{\ell_2}(k_2,\eta_0)P(k_1)P(k_2) \\
& (-i)^{\ell_1+\ell_2+\ell''+\ell_2''} i^{\ell_1'+\ell_2'+\ell_1''}(2\ell_1'+1)(2\ell_2'+1)(2\ell_1''+1)(2\ell_2''+1)(2\ell''+1) \\
& \begin{pmatrix} \ell_1' & \ell_2' & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_2 & \ell_2' & \ell_2'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1' & \ell_1 & \ell_2'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1'' & \ell_2'' & \ell'' \\ 0 & 0 & 0 \end{pmatrix}^2 \\
& j_{\ell_1'}\left[k_1(\eta-\eta_0)\right] j_{\ell_2'}\left[k_2(\eta-\eta_0)\right] j_{\ell_1''}\left[k_1(\eta'-\eta)\right] j_{\ell_2''}\left[k_2(\eta'-\eta)\right] + \text{permutation}
\end{aligned}
\tag{B4}
$$

Note that this solution is approximate but does not assume tight coupling, despite the fact that we used the tight coupling solutions Equations 29, 33 and 34 as a trial solution. Equation B4 is the result of iterating the integral equation once and will therefore contain corrections beyond the tight coupling approximation. In particular this solution takes into account all the terms in the full Boltzmann hierarchy, Equations 17-19. The dominant contribution would come from around the last scattering surface, that is when $\eta' - \eta \sim 0$. In that case the corresponding spherical Bessel functions would be close to zero unless the order of the spherical Bessel function is zero. Thus we would expect that most contribution

comes from terms with $\ell_1'' = \ell_2'' = 0$. The last Wigner 3jm symbol then forces $\ell'' = 0$. But $f_{\ell''=0} = 0$ since the first order monopole cancels out making $S^{m_1 m_2 m_3}_{\ell_1 \ell_2 \ell_3}$ vanish. This is the result that we found for the approximate solution of the second order Boltzmann equations also. For $\ell_1'', \ell_2'' \neq 0$ we also note that the arguments of the first two spherical Bessel functions differ from the arguments of the last two spherical Bessel functions by a factor of ~ 100. But for the squeezed triangles we would expect either ℓ_1 or ℓ_2 to be small making $\ell_1' \sim \ell_2''$ or $\ell_2' \sim \ell_2''$ due to triangle conditions in Wigner 3jm symbols. Thus we have a product of the spherical Bessel functions of similar orders but with arguments differing by a factor of hundred. This product will be negligibly small, since if one of the spherical Bessel function is near the peak the other would be negligibly small or oscillating very fast giving a small residual after integration. Thus the contribution from the second order monopole can be safely neglected for the case of inhomogeneous recombination. This argument also applies to all other terms in the second order Boltzmann equation which are a product of monopole type term and higher order multipoles.